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A PRIMARY CLASSIFICATION OF PROJECTIVE TRANSFORMATIONS IN FUNCTION SPACE.*

BY L. L. DINES.

In an earlier paper† we have defined and studied the projective functional transformation

$$\phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy}{\delta + \int_0^1 \epsilon(y)\phi(y)dy}, \tag{1}$$

represented symbolically by the array of coefficients

$$\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}$$

Obviously the transformation (1) includes as a special case the Fredholm transformation

$$\phi'(x) = \phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy.$$

It also includes other transformations which may properly be said to be *projectively equivalent* to Fredholm transformations. One of the objects of the present paper is to characterize this class of transformations.

Such projective transformations as do not fall into the above class are themselves divided into three classes on the basis of projective equivalence. The classification is accomplished by a consideration of invariant elements of two kinds: *points* and *lineoids*, the lineoid being the natural dual of the point in function space.

§ 1. HOMOGENEOUS COÖRDINATES.

By virtue of its form, (1) transforms every point $\phi(x)$ of function space $\mathfrak{C}_1$ into a point $\phi'(x)$ of the same space, with the exception of those points defined by the equation

$$\delta + \int_0^1 \epsilon(y)\phi(y)dy = 0.$$

With a view to attaining a domain which is closed under the projective transformation we introduce homogeneous coördinates. To this end, we make in (1) the substitutions

$$\phi(x) = \omega_1(x)/\omega_2, \qquad \phi'(x) = \omega_1'(x)/\omega_2', \tag{2}$$

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* Read under slightly different title before the American Mathematical Society, September 3, 1919.

† *Transactions of the American Mathematical Society*, Vol. 20, pages 45–65. This paper will be referred to as *Proj. Trans.*

with the understanding that

$$\omega_1(x) \textit{ and } \omega_1'(x) \textit{ belong to } \mathfrak{C}_1,$$
$$\omega_2 \quad \textit{and} \quad \omega_2' \quad \textit{belong to } \mathfrak{C}_0',$$

where $\mathfrak{C}_0'$ denotes the class of real constants with 0 excluded.

The resulting transformation is equivalent to the linear *homogeneous* transformation

$$\begin{aligned} r\omega_1'(x) &= \beta(x)\omega_1(x) + \int_0^1 \gamma(xy)\omega_1(y)dy + \alpha(x)\omega_2 \\ r\omega_2' &= \int_0^1 \epsilon(y)\omega_1(y)dy + \delta\omega_2 \end{aligned} \tag{3}$$

where r is a factor of proportionality.

This homogeneous transformation operates upon the binary elements

$$(\omega_1(x),\ \omega_2)$$

of the composite class $\mathfrak{C}_1\mathfrak{C}_0$, transforming each into a binary element

$$(r\omega_1'(x),\ r\omega_2')$$

of the same class. If ω_2 and ω_2' are restricted to be different from zero, (3) is equivalent to (1) in the light of the substitution (2). Nothing inherent in the form of the homogeneous transformation (3) however demands this restriction.

Following the suggestion dictated by analogy with ordinary projective geometry, we will interpret the binary element $(\omega_1(x),\ \omega_2)$ in which *either one* of the constituent elements is different from zero, as the homogeneous coördinates of a point in function space, it being understood that $(\omega_1(x),\ \omega_2)$ and $(\omega_1'(x),\ \omega_2')$ represent the same point, if and only if there is a constant r such that $\omega_1 = r\omega_1'$ and $\omega_2 = r\omega_2'$.

The class

$$\overline{\mathfrak{C}_1\mathfrak{C}_0} \equiv [\textit{all points } (\omega_1(x),\ \omega_2')]$$

will constitute the *projective function space*, and the transformation (3) the projective functional transformation in homogeneous coördinates.

The results obtained in *Proj. Trans.* can easily be translated into terms of homogeneous coördinates. For example the inverse of (3), in case the determinant B is different from zero, is

$$\begin{aligned} r'\omega_1(x) &= \beta'(x)\omega_1'(x) + \int_0^1 \gamma'(xy)\omega_1'(y)dy + \alpha'(x)\omega_2', \\ r'\omega_2 &= \int_0^1 \epsilon'(y)\omega_1'(y)dy + \delta'\omega_2', \end{aligned} \tag{3'}$$

where the coefficients $\alpha',\ \beta',\ \gamma',\ \delta',\ \epsilon'$ have the same definitions as in *Proj. Trans.* (8).

§ 2. THE LINEOID, AND DUALITY.

The class of points $(\omega_1(x),\ \omega_2)$ satisfying an equation of form

$$\int_0^1 v_1(y)\omega_1(y)dy + v_2\omega_2 = 0 \tag{4}$$

where v_1 belongs to $\mathfrak{C}_1$ and v_2 to $\mathfrak{C}_0$ and one at least is different from zero, will be called a *lineoid* in function space.

Since the lineoid (4) is characterized by the pair of coefficients

$$(v_1(y),\ v_2),$$

this pair of coefficients will serve as coördinates of the lineoid, evidently homogeneous coördinates, since the lineoids $(v_1(y),\ v_2)$ and $(v_1'(y),\ v_2')$ coincide if and only if there is a constant r such that $v_1(y) = rv_1'(y)$ and $v_2 = rv_2'$.

The equation (4) expresses the dualistic relation of conjointness between a point $(\omega_1(x),\ \omega_2)$ and a lineoid $(v_1(y),\ v_2)$ which simultaneously satisfy it.

Suppose now that every point of a given lineoid (4) be subjected to a projective transformation (3). What will be the locus of the transformed points $(\omega_1'(x),\ \omega_2')$? To ascertain this we introduce the value of $(\omega_1(x),\ \omega_2)$ in terms of $(\omega_1'(x),\ \omega_2')$ from (3′) into (4), and obtain an equation which reduces to the form

$$\int_0^1 v_1'(y)\omega_1'(y) + v_2'\omega_2' = 0,$$

where

$$\begin{aligned} rv_1'(y) &= v_1(y)\beta'(y) + \int_0^1 v_1(x)\gamma'(xy)dx + v_2\epsilon'(y), \\ rv_2' &= \int_0^1 v_1(x)\alpha'(x)dx + v_2\delta'. \end{aligned} \tag{5}$$

This is by definition the equation of a lineoid. Hence, *the projective transformation* (3) *transforms every lineoid into a lineoid.* More explicitly, *the lineoid* $(v_1(y),\ v_2)$ *is transformed by the projective transformation* (3) *into a lineoid* $(v_1'(y),\ v_2')$ *whose coördinates are given by* (5). Also, *a point and a lineoid which are conjoint are transformed by a projective transformation into a point and a lineoid which are conjoint.*

§ 3. EXPRESSION OF THE PROJECTIVE TRANSFORMATION IN THE FORM OF A FREDHOLM TRANSFORMATION.

It has recently been pointed out by Hildebrandt* that the transformation (1) can be expressed in the form of a Fredholm transformation, operative upon functions on a suitably defined range. The procedure used in this section is essentially that suggested by Hildebrandt.

By our conventions in *Proj. Trans.*, the range of the argument in the functions of $\mathfrak{C}_1$ is the class of real numbers from 0 to 1, inclusive. Let us denote this range by $\mathfrak{J}_1$; that is

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* *Bulletin of the American Mathematical Society*, Vol. 26 (1920), page 400.

$$\mathfrak{J}_1 \equiv [\textit{all real numbers from } 0 \textit{ to } 1, \textit{ inclusive}].$$

Let us likewise define

$$\mathfrak{J}_2 \equiv [\textit{all real numbers from } 1 \textit{ to } 2, \textit{ including } 2],$$
$$\mathfrak{J}_1 + \mathfrak{J}_2 \equiv [\textit{all real numbers from } 0 \textit{ to } 2, \textit{ inclusive}].$$

By composition of these ranges we get the following composite ranges (each represented geometrically by the points of a square):

$$\mathfrak{J}_1\mathfrak{J}_1, \quad \mathfrak{J}_1\mathfrak{J}_2, \quad \mathfrak{J}_2\mathfrak{J}_1, \quad \mathfrak{J}_2\mathfrak{J}_2, \quad (\mathfrak{J}_1 + \mathfrak{J}_2)(\mathfrak{J}_1 + \mathfrak{J}_2).$$

Now let us suppose the binary elements $(\omega_1(x), \omega_2)$ and $(\omega_1'(x), \omega_2')$ to be replaced respectively by the binary elements $(\omega_1(x), \omega_2(x))$ and $(\omega_1'(x), \omega_2'(x))$, with the understanding that $\omega_2(x)$ and $\omega_2'(x)$ are constant functions having the definitions:

$$\omega_2(x) = \omega_2, \qquad \omega_2'(x) = \omega_2', \qquad x \textit{ on } \mathfrak{J}_2.$$

Then the equations of (3) are equivalent to

$$\begin{aligned} r\omega_1'(x) &= \beta(x)\omega_1(x) + \int_0^1 \gamma(xy)\omega_1(y)dy + \int_1^2 \alpha(x)\omega_2(y)dy, \\ r\omega_2'(x) &= \omega_2(x) + \int_0^1 \epsilon(y)\omega_1(y)dy + \int_1^2 (\delta - 1)\omega_2(y)dy, \end{aligned} \tag{6}$$

and by the classic method of Fredholm* for reducing a system of integral equation to a single equation, this pair of equations can be represented by

$$r\rho(x)\omega'(x) = \omega(x) + \int_0^2 \kappa(xy)\omega(y)dy, \qquad x \textit{ on } \mathfrak{J}_1 + \mathfrak{J}_2, \tag{7}$$

where

$$\omega(x) = \begin{cases} \omega_1(x) & \textit{on } \mathfrak{J}_1 \\ \omega_2 & \textit{on } \mathfrak{J}_2 \end{cases} \qquad \kappa(x, y) = \begin{cases} \gamma(x, y)/\beta(x) & \textit{on } \mathfrak{J}_1\mathfrak{J}_1 \\ \alpha(x)/\beta(x) & \textit{on } \mathfrak{J}_1\mathfrak{J}_2 \\ \epsilon(y) & \textit{on } \mathfrak{J}_2\mathfrak{J}_1 \\ \delta - 1 & \textit{on } \mathfrak{J}_2\mathfrak{J}_2 \end{cases}$$

$$\omega'(x) = \begin{cases} \omega_1'(x) & \textit{on } \mathfrak{J}_1 \\ \omega_2' & \textit{on } \mathfrak{J}_2 \end{cases}$$

$$\rho(x) = \begin{cases} 1/\beta(x) & \textit{on } \mathfrak{J}_1 \\ 1 & \textit{on } \mathfrak{J}_2 \end{cases}$$

The transformation (7) is an ordinary Fredholm transformation, transforming a function $\omega(x)$ on $\mathfrak{J}_1 + \mathfrak{J}_2$ into a function $r\rho(x)\omega'(x)$ on the same interval. The kernel $\kappa(xy)$ is continuous on the square $(\mathfrak{J}_1 + \mathfrak{J}_2)(\mathfrak{J}_1 + \mathfrak{J}_2)$ except for the lines of possible discontinuity $x = 1$ and $y = 1$.

In a similar manner, the lineoid transformation (5) can be replaced by

$$r\rho'(y)v'(y) = v(y) + \int_0^2 v(x)\kappa'(x, y)dx, \qquad y \textit{ on } \mathfrak{J}_1 + \mathfrak{J}_2, \tag{7'}$$

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* Acta mathematica, Vol. 27.

where

$$v(y) = \begin{cases} v_1(y) & on\,\mathfrak{I}_1 \\ v_2 & on\ \mathfrak{I}_2 \end{cases}$$

$$v'(y) = \begin{cases} v_1(y) & on\ \mathfrak{I}_1 \\ v_2' & on\ \mathfrak{I}_2 \end{cases} \qquad \kappa'(x, y) = \begin{cases} \gamma'(x, y)/\beta'(y) & on\ \mathfrak{I}_1\mathfrak{I}_1 \\ \epsilon'(y)/\beta'(y) & on\ \mathfrak{I}_2\mathfrak{I}_1 \\ \alpha'(x) & on\ \mathfrak{I}_1\mathfrak{I}_2 \\ \delta' - 1 & on\ \mathfrak{I}_2\mathfrak{I}_2 \end{cases}$$

$$\rho'(y) = \begin{cases} 1/\beta'(y) & on\ \mathfrak{I}_1 \\ \ \ 1 & on\ \mathfrak{I}_2 \end{cases}$$

The kernels $\kappa(x, y)$ and $\kappa'(x, y)$ of the transformations (7) and (7′) satisfy the two identities

$$\kappa(x, y) + \kappa'(x, y) + \int_0^2 \kappa(x, z)\kappa'(z, y)dz = 0$$
$$\kappa'(x, y) + \kappa(x, y) + \int_0^2 \kappa'(x, z)\kappa(z, y)dz = 0$$

on the square $(\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2)$, as may be verified by use of the relations (9) and (9′) of *Proj. Trans.* That is, they are *reciprocal kernels* in the Fredholm sense.

§ 4. INVARIANT POINTS AND LINEOIDS.

We now inquire as to what points, if any, are transformed into themselves by (3), restricting ourselves for the present however to the important special case in which

$$(8) \qquad \beta(x) = 1.$$

From (7) which is equivalent to (3), we see that under the condition (8), the equation for such invariant points is

$$(9) \qquad (1 - r)\omega(x) + \int_0^2 \kappa(x, y)\omega(y)dy = 0, \qquad x\ on\ \mathfrak{I}_1 + \mathfrak{I}_2.$$

Every non-zero solution $\omega(x)$ of this equation determines a binary element $(\omega_1(x), \omega_2(x))$ in which

$$\omega_1(x) = \omega(x) \qquad on\ \mathfrak{I}_1,$$
$$\omega_2(x) = \omega(x) \qquad on\ \mathfrak{I}_2.$$

If $\omega_2(x)$ is constant, this binary element represents an invariant point. And $\omega_2(x)$ cannot be other than a constant except in the single case $r = 1$, as will be evident upon consideration of the content of equation (9) when x is on $\mathfrak{I}_2$, namely

$$(10) \qquad (1 - r)\omega_2(x) + \int_0^1 \epsilon(y)\omega_1(y)dy + \int_0^2 (\delta - 1)\omega_2(y)dy = 0.$$

In case $r = 1$, a solution $\omega = (\omega_1(x), \bar{\omega}_2(x))$ may exist in which $\bar{\omega}_2(x)$ is not constant, but corresponding to every such solution there will be a solution $\omega = (\omega_1(x), \omega_2)$ in which $\omega_2 = \int_0^1 \bar{\omega}_2(y)dy$. This solution will represent an invariant point unless $\omega_1(x) = \omega_2 = 0$.

Equation (9) is an homogeneous Fredholm integral equation containing the parameter r. The non-zero solutions $\omega(x)$ corresponding to values of r different from 1 are the so-called *fundamental functions in* x for the kernel $\kappa(x, y)$. If we denote by the term *asymptotic fundamental functions* those non-zero solutions $\omega(x)$ corresponding to the parameter value $r = 1$, we may state our conclusions in the form

THEOREM I: *If* $\beta(x) = 1$, *the invariant points* $\omega(x) = (\omega_1(x), \omega_2)$ *of the transformation* (3) *are precisely the ordinary fundamental functions in* x *of the kernel* $\kappa(x, y)$, *together with those asymptotic fundamental functions in* x *which are constant on the interval* $\mathfrak{J}_2$.

By an analogous consideration of (7′), we obtain

THEOREM I′: *If* $\beta(x) = 1$, *the invariant lineoids* $v(y) = (v_1(y), v_2)$ *of the transformation* (3) *are precisely the ordinary fundamental functions in* y *of the kernel* $\kappa'(x, y)$, *together with those asymptotic fundamental functions in* y *which are constant on the interval* $\mathfrak{J}_2$.

In this theorem the invariant lineoids are characterized by means of the reciprocal kernel $\kappa'(x, y)$. They are characterized directly in terms of $\kappa(x, y)$ in the following

COROLLARY: *If* $\beta(x) = 1$, *the invariant lineoids of the transformation* (3) *are precisely the ordinary fundamental functions in* y *of the kernel* $k(x, y)$, *together with those asymptotic fundamental functions which are constant on the interval* $\mathfrak{J}_2$.

To prove this we need only show that the solutions of the two equations

$$(1 - r')v(y) + \int_0^2 v(x)\kappa'(x, y)dx = 0 \tag{11′}$$

and

$$(1 - r)v(y) + \int_0^2 v(x)\kappa(x, y)dx = 0 \tag{11}$$

coincide.

Suppose $\bar{v}(y)$ is a solution of (11′). Then for some value of r',

$$r'\bar{v}(y) = \bar{v}(y) + \int_0^2 \bar{v}(x)\kappa'(x, y)dx. \tag{12}$$

We multiply equation (11) by r' and write it in the form

$$r'rv(y) = r'v(y) + r'\int_0^2 v(x)\kappa(x, y)dx. \tag{13}$$

Upon substituting in (13) $\bar{v}$ for v in the left member and the value of $r'\bar{v}$ from (12) for $r'v$ in the right member, we obtain after some rearrangement

$$r'r\bar{v}(y) = \bar{v}(y) + \int_0^2 \bar{v}(x)\left[\kappa'(x, y) + \kappa(x, y) + \int_0^2 \kappa'(x, z)\kappa(z, y)dz\right]dx.$$

Since $\kappa(x, y)$ and $\kappa'(x, y)$ are reciprocal kernels this reduces to

$$r'r\bar{v}(y) = \bar{v}(y),$$

which proves that $\bar{v}(y)$ is a solution of (11) corresponding to the parameter value $r = 1/r'$.

In a similar manner it can be shown that each solution of (11) is a solution of (11').

§ 5. SYMMETRIC PROJECTIVE TRANSFORMATIONS.

The theorems of the preceding section suggest interesting properties of those projective transformations for which the corresponding kernels $\kappa(x, y)$ are symmetric. The most general such transformation is of form

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2, \\ r\omega_2' &= \int_0^1 \alpha(y)\omega_1(y)dy + \delta\omega_2, \end{aligned}$$

where $\gamma(x, y) = \gamma(y, x)$.

Transformations of this form will be called *symmetric projective transformations*.

From the well-known theory of integral equations with symmetric kernels,* we have by virtue of the theorems of the preceding section

THEOREM II. *Every symmetric projective transformation has at least one invariant point, and one invariant lineoid, which are not conjoint.*

The existence of the invariant elements follows from the fact that every symmetric kernel has a characteristic value; the non-conjointness from the fact that every such characteristic value is a simple pole of the resolvent, and hence (see Goursat, loc. cit., page 411) for every fundamental function ω in x of the kernel $\kappa(x, y)$ there is a fundamental function v in y such that

$$\int_0^2 v(y)\omega(y)dy \neq 0,$$

that is

$$\int_0^1 v_1(y)\omega_1(y)dy + v_2\omega_2 \neq 0.$$

§ 6. PROJECTIVE TRANSFORMATIONS WITHOUT INVARIANT POINTS AND LINEOIDS.

While, as we have just seen, every symmetric projective transformation admits an invariant point and lineoid, such is not the case with non-symmetric transformations. In this section we present four examples of transformations qualified respectively as follows:

1. Admitting an invariant lineoid, but no invariant point.
2. Admitting an invariant point, but no invariant lineoid.
3. Admitting neither invariant point nor lineoid.
4. Admitting an invariant point and an invariant lineoid, but no pair which are not conjoint.

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* See, for instance, Goursat's *Cours d'analyse mathematique*, Vol. III, § 587.

We shall make use of the following

LEMMA:* If the traces

$$A_n = \int_0^2\int_0^2 \cdots \int_0^2 \kappa(s_1, s_2)\kappa(s_2, s_3) \cdots \kappa(s_n, s_1)ds_1ds_2 \cdots ds_n$$

of the kernel $\kappa(x, y)$ are zero for all values of n greater than or equal to 3, then $\kappa(x, y)$ admits no characteristic value, and hence no ordinary fundamental function.

We take now any complete† normalized orthogonal system of continuous functions ϕ_n, $n = 1, 2, \cdots$, on the interval $\mathfrak{J}_1$.

From this system which is orthogonal on $\mathfrak{J}_1$ we form a system Φ_n, $n = 0, 1, 2, \cdots$, orthogonal on $\mathfrak{J}_1 + \mathfrak{J}_2$, as follows:

$$\Phi_0 \equiv [0 \text{ } on \text{ } \mathfrak{J}_1, 1 \text{ } on \text{ } \mathfrak{J}_2],$$
$$\Phi_n \equiv [\phi_n \text{ } on \text{ } \mathfrak{J}_1, 0 \text{ } on \text{ } \mathfrak{J}_2], \qquad n = 1, 2, \cdots.$$

Our examples will be constructed in terms of the functions $\{\phi_n\}$; and in each case the kernel $\kappa(x, y)$ to which the transformation gives rise will be expansible in terms of the functions $\{\Phi_n\}$.

Example 1. Consider the transformation

$$\text{(14)} \qquad \begin{pmatrix} 1+\gamma & \alpha \\ 0 & 1 \end{pmatrix}$$

in which

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n\phi_{n+1}(x)\phi_n(y), \qquad \alpha(x) \equiv \phi_1(x),$$

the coefficients c_n being all distinct from zero and so chosen that the series for $\gamma(x, y)$ converges uniformly on the square $\mathfrak{J}_1\mathfrak{J}_1$.

The corresponding kernel $\kappa(x, y)$ can be written in the form

$$\kappa(x, y) \equiv \sum_{n=0}^{\infty} c_n\Phi_{n+1}(x)\Phi_n(y) \quad on \ (\mathfrak{J}_1 + \mathfrak{J}_2)(\mathfrak{J}_1 + \mathfrak{J}_2),$$

where $c_0 = 1$.

The traces of this kernel $\kappa(x, y)$ are all zero. Hence by the lemma the kernel admits no singular values, and no ordinary fundamental functions.

Nor does it admit any asymptotic fundamental functions which can furnish invariant points. For the equation

$$\int_0^2 \kappa(x, y)\omega(y)dy = 0$$

is equivalent to

$$\sum_{n=0}^{\infty} c_n\Phi_{n+1}(x)\int_0^2\Phi_n(y)\omega(y)dy = 0.$$

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[*] Cf. Goursat, loc. cit., page 428.

[†] An orthogonal system of functions is said to be *complete* if there is no function which is orthogonal to all the functions of the system, and such that the integral of its square is 1. Cf. Goursat, loc. cit., pages 445 and 446.

Multiplying this equation by $\Phi_{k+1}(x)$ and integrating with respect to x, we get, on account of the orthogonality,

$$c_k \int_0^2 \Phi_k(y)\omega(y)dy = 0, \qquad k = 0, 1, 2, \cdots.$$

That is, since $c_k \neq 0$, $\omega(y)$ must be orthogonal to all the functions $\Phi_k(y)$, on the interval $\mathfrak{J}_1 + \mathfrak{J}_2$.

But from the definitions of the Φ's, it is clear that there can be no such function ω which is constant on $\mathfrak{J}_2$. Hence, *the transformation*

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2, \\ r\omega_2' &= \omega_2, \end{aligned}$$

represented by (14), *admits no invariant point. It admits the invariant lineoid* $\omega_2 = 0$.

Example 2. *The transformation*

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy, \\ r\omega_2' &= \omega_2 \quad + \int_0^1 \epsilon(y)\omega_1(y)dy, \end{aligned}$$

in which

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n\phi_n(x)\phi_{n+1}(y), \qquad \epsilon(y) \equiv \phi_1(y),$$

has no invariant lineoid, but admits the invariant point (0, 1).

The kernel $\kappa(x, y)$ for this transformation can be written in the form

$$\kappa(x, y) = \sum_{n=0}^{\infty} c_n\Phi_n(x)\Phi_{n+1}(y),$$

from which fact the properties of the transformation follow as in Example 1.

Example 3. Consider the transformation whose coefficients are*

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n[\phi_{2n+1}(x)\phi_{2n-1}(y) + \phi_{2n}(x)\phi_{2n+2}(y)], \tag{15}$$
$$\alpha(x) \equiv \phi_1(x), \qquad \epsilon(y) \equiv \phi_2(y), \qquad \delta \equiv 1.$$

The corresponding kernel $\kappa(x, y)$ can be written in the form $\kappa(x, y) = \Phi_1(x)\Phi_0(y) + \Phi_0(x)\Phi_2(y)$

$$\sum_{n=1}^{\infty} c_n[\Phi_{2n+1}(x)\Phi_{2n-1}(y) + \Phi_{2n}(x)\Phi_{2n+2}(y)].$$

All the traces of this kernel are zero; hence it has no ordinary fundamental functions. Nor does it admit asymptotic fundamental functions in either x or y which are constant on $\mathfrak{J}_2$, as may be shown by the method used in Example 1. Hence *the transformation defined by* (15) *admits no invariant element.*

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* The form of this transformation was suggested by an example kindly furnished me by Professor E. W. Chittenden.

Example 4. Consider the transformation

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2, \\ r\omega_2' &= \omega_2, \end{aligned}$$

where

$$\gamma(x, y) \equiv \sum_{n=2}^{\infty} c_n \phi_{n+1}(x)\phi_n(y), \qquad \alpha(x) \equiv \phi_1(x).$$

The corresponding kernel $\kappa(x, y)$ given by

$$\kappa(x, y) = \Phi_1(x)\Phi_0(y) + \sum_{n=2}^{\infty} c_n \Phi_{n+1}(x)\Phi_n(y)$$

has no ordinary fundamental functions since all of its traces are zero.

The transformation admits a single invariant point given by the asymptotic fundamental function

$$\omega(x) = \Phi_1(x) = (\phi_1(x), 0),$$

while it admits the two invariant lineoids

$$v(y) = \Phi_0(y) = (0, 1), \qquad v(y) = \Phi_2(y) = (\phi_2(y), 0).$$

It can be verified immediately that *the invariant point is on both of the invariant lineoids.*

§ 7. REDUCTION TO CANONICAL FORM.

Two transformations

$$\bar{S} \equiv \left(\begin{array}{ccc} \bar{\beta} + & \bar{\gamma} & \bar{\alpha} \\ & \bar{\epsilon} & \bar{\delta} \end{array}\right), \qquad \bar{\bar{S}} \equiv \left(\begin{array}{ccc} \bar{\bar{\beta}} + & \bar{\bar{\gamma}} & \bar{\bar{\alpha}} \\ & \bar{\bar{\epsilon}} & \bar{\bar{\delta}} \end{array}\right)$$

are said to be projectively equivalent if there exists a non-singular projective transformation T with inverse T^{-1}:

$$T \equiv \left(\begin{array}{ccc} \beta + & \gamma & \alpha \\ & \epsilon & \delta \end{array}\right), \qquad T^{-1} \equiv \left(\begin{array}{ccc} \beta' + & \gamma' & \alpha' \\ & \epsilon' & \delta' \end{array}\right),$$

such that

$$T\bar{S}T^{-1} = \bar{\bar{S}}. \tag{16}$$

The relation between $\bar{S}$ and $\bar{\bar{S}}$ is evidently a reciprocal one, since from (16) it follows that $T^{-1}\bar{\bar{S}}T = \bar{S}$.

Suppose now that $\bar{S}$ admits the invariant point $\bar{\omega} \equiv (\bar{\omega}_1(x), \bar{\omega}_2)$. Then $\bar{\bar{S}}$ admits as invariant point the transform of $\bar{\omega}$ by T. For from the equation

$$\bar{S}\bar{\omega} = r\bar{\omega},$$

which expresses the fact that $\bar{\omega}$ is invariant under $\bar{S}$, there follows by use of (16)

$$\begin{aligned}\bar{\bar{S}}(T\bar{\omega}) &= T\bar{S}T^{-1}(T\bar{\omega})\\ &= T\bar{S}\bar{\omega}\\ &= Tr\bar{\omega}\\ &= r(T\bar{\omega}),\end{aligned}$$

which expresses the fact that $T\bar{\omega}$ is invariant under $\bar{\bar{S}}$.

Likewise, if $\bar{S}$ admits the invariant lineoid $\bar{v}$, then $\bar{\bar{S}}$ admits as invariant lineoid the transform of $\bar{v}$ by T.

It is evident from these remarks that a *necessary* condition for projective equivalence is a one-to-one correspondence between the invariant elements of two transformations. This condition is not sufficient, and we shall not undertake here the general question of sufficient conditions.

We shall show, however, that a projective transformation which admits *any* invariant element is projectively equivalent to a transformation of one of the three forms

$$(17)\qquad \begin{pmatrix}\beta+\gamma & 0\\ \epsilon & \delta\end{pmatrix},\quad \begin{pmatrix}\beta+\gamma & \alpha\\ 0 & \delta\end{pmatrix},\quad \begin{pmatrix}\beta+\gamma & 0\\ 0 & \delta\end{pmatrix},$$

of which the first admits the invariant point (0, 1), the second admits the invariant lineoid (0, 1), while the third admits both of these invariant elements.

We shall need to make use of the explicit forms of the coefficients of $\bar{\bar{S}}$ in terms of the coefficients of T, $\bar{S}$, and T^{-1} as determined by (16). These can easily be written down by use of the formulas obtained in *Proj. Trans.*, § 5. In terms of the abbreviated notations there used, we have

$$(18)\qquad \begin{aligned}\bar{\bar{\beta}} &= \beta\bar{\beta}\beta' = \bar{\beta},\\ \bar{\bar{\gamma}} &= \beta\bar{\beta}\gamma' + \beta\bar{\gamma}\beta' + \gamma\bar{\beta}\beta' + (J\gamma\bar{\gamma})\beta' + \alpha\cdot\bar{\epsilon}\beta' + \beta J\bar{\gamma}\gamma'\\ &\qquad + J\gamma\bar{\beta}\gamma' + JJ\gamma\bar{\gamma}\gamma' + \alpha J\bar{\epsilon}\gamma' + \beta J\bar{\alpha}\epsilon' + JJ\gamma\bar{\alpha}\epsilon' + J\alpha\bar{\delta}\epsilon',\\ \bar{\bar{\alpha}} &= \beta(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + J\gamma(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + \alpha(J\bar{\epsilon}\alpha' + \bar{\delta}\delta'),\\ \bar{\bar{\epsilon}} &= (\epsilon\bar{\beta} + J\epsilon\bar{\gamma} + \delta\bar{\epsilon})\beta' + J(\epsilon\bar{\beta} + J\epsilon\bar{\gamma} + \delta\bar{\epsilon})\gamma' + (J\epsilon\bar{\alpha} + \delta\bar{\delta})\epsilon',\\ \bar{\bar{\delta}} &= J\epsilon(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + (J\epsilon\bar{\alpha} + \delta\bar{\delta})\delta'.\end{aligned}$$

Suppose now that $\bar{S}$ admits the invariant point $(\bar{\omega}_1, \bar{\omega}_2)$, that is

$$(19)\qquad \begin{aligned} r\bar{\omega}_1 &= \bar{\beta}\bar{\omega}_1 + J\bar{\gamma}\bar{\omega}_1 + \bar{\alpha}\bar{\omega}_2,\\ r\bar{\omega}_2 &= \qquad\qquad J\bar{\epsilon}\bar{\omega}_1 + \bar{\delta}\bar{\omega}_2.\end{aligned}$$

Then if T be so chosen that

$$(\alpha', \delta') \equiv (\bar{\omega}_1, \bar{\omega}_2),$$

the value of $\bar{\bar{\alpha}}$ as determined by (18) may in view of (19) be written

$$\bar{\bar{\alpha}} = r(\beta\alpha' + J\gamma\alpha' + \alpha\delta').$$

Hence

$$\bar{\bar{\alpha}} = 0,$$

on account of the relations between the coefficients of T and T^{-1} (see (9′) of *Proj. Trans.*, I); and the transformation $\bar{\bar{S}}$ projectively equivalent to $\bar{S}$ is of the form occurring first in (17).

Similarly, it may be seen that if $\bar{S}$ admits the invariant lineoid $(\bar{v}_1, \bar{v}_2)$, and if T be so chosen that

$$(\epsilon, \delta) \equiv (\bar{v}_1, \bar{v}_2),$$

then

$$\bar{\bar{\epsilon}} = 0;$$

that is, the transformation $\bar{\bar{S}}$ is of the form appearing second in (17).

The third transformation in (17) admits the invariant point (0, 1) and the invariant lineoid (0, 1)—two elements which are not conjoint. Hence no transformation can be equivalent to it unless it admits two similarly related invariant elements. Let us then suppose that the transformation $\bar{S}$ admits the invariant point $(\bar{\omega}_1, \bar{\omega}_2)$ and the invariant lineoid $(\bar{v}_1, \bar{v}_2)$, and that these two elements are not conjoint, that is

$$J\bar{\omega}_1\bar{v}_1 + \bar{\omega}_2\bar{v}_2 = 1. \tag{20}$$

We shall show that under these conditions, T may be so chosen that $\bar{\bar{S}}$ is of the form occurring third in (17). For this purpose it will be sufficient, in view of the two cases just preceding, to exhibit a transformation T with inverse T^{-1} satisfying the *two* conditions

$$(\epsilon, \delta) \equiv (\bar{v}_1, \bar{v}_2), \qquad (\alpha', \delta') \equiv (\bar{\omega}_1, \bar{\omega}_2).$$

We may on account of (20) and the homogeneity of coördinates assume without loss of generality that *if* $J\bar{\omega}_1\bar{v}_1 = 0$, *then* $\bar{\omega}_2 = \bar{v}_2 = 1$. This will simplify the discussion.

Consider the transformation

$$T = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}$$

in which

$$\beta \equiv 1, \qquad \gamma \equiv c\bar{\omega}_1 \cdot \bar{v}_1, \qquad \alpha \equiv -\bar{\omega}_1, \qquad \epsilon \equiv \bar{v}_1, \qquad \delta \equiv \bar{v}_2, \tag{21}$$

c being a constant to be determined.

The coefficients α', δ' of the inverse transformation T^{-1} are given by the formulas*

$$\alpha'(x) = A(x)/B, \qquad \delta' = D/B,$$

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* *Proj. Trans.*, (8).

where for the particular transformation T defined by (21)

$$A(x) = \bar{\omega}_1(x), \qquad D = 1 + cJ\bar{\omega}_1\bar{v}_1,$$
$$B = \bar{v}_2(1 + cJ\bar{\omega}_1 v_1) + J\bar{\omega}_1\bar{v}_1.$$

We now choose c so that

$$D = 1 + cJ\bar{\omega}_1\bar{v}_1 = \bar{\omega}_2.$$

This can obviously be done if $J\bar{\omega}_1\bar{v}_1 \neq 0$, while if $J\bar{\omega}_1\bar{v}_1 = 0$, then $D = 1 = \bar{\omega}_2$ by our assumption. With this choice of c, B becomes by (20) equal to unity. Hence

$$\alpha'(x) = \bar{\omega}_1(x), \qquad \delta' = \bar{\omega}_2,$$

and the transformation defined by (21) possesses the desired properties.

Our results may be stated in the

THEOREM IV: *A projective transformation is projectively equivalent to transformations of the following forms, under the conditions respectively specified, and only under those conditions:*

$$\begin{pmatrix} \beta + \gamma & 0 \\ \epsilon & \delta \end{pmatrix} \textit{ if it admits an invariant point,}$$

$$\begin{pmatrix} \beta + \gamma & \alpha \\ 0 & \delta \end{pmatrix} \textit{ if it admits an invariant lineoid,}$$

$$\begin{pmatrix} \beta + \gamma & 0 \\ 0 & \delta \end{pmatrix} \textit{ if it admits an invariant lineoid and an invariant point not on it.}$$

Upon the basis of this theorem, we are able to classify all projective transformations under the following four types:

Type I, Transformations admitting an invariant lineoid and an invariant point not on it; projectively equivalent to the form

$$\text{I.} \qquad \begin{pmatrix} \beta + \gamma & 0 \\ 0 & \delta \end{pmatrix}.$$

Type II, Transformations admitting an invariant point, but not belonging to Type I; projectively equivalent to the form

$$\text{II.} \qquad \begin{pmatrix} \beta + \gamma & 0 \\ \epsilon & \delta \end{pmatrix},$$

but not to the form I.

Type III, Transformations admitting an invariant lineoid, but not belonging to Type I; projectively equivalent to the form

$$\text{III.} \qquad \begin{pmatrix} \beta + \gamma & \alpha \\ 0 & \delta \end{pmatrix},$$

but not to the form I.

Type IV, Transformations admitting no invariant element; projectively equivalent to no transformation of form I, II, or III.

Each of the four types defines a class of transformations which contains at least one member. Types I and IV define classes which do not overlap each other or either of the remaining classes. The classes defined by Types II and III do overlap, as is shown by Example 4, of § 6.

§ 8. SIMPLIFICATION OF THE TYPE FORMS BY FACTORIZATION.

From the first equation of (18) it is clear that the coefficient $\bar{\beta}$ of a transformation $\bar{S}$ is unchanged when $\bar{S}$ undergoes any projective transformation whatever. Any projective transformation may however be *factored* into two transformations of which one is of a very simple type, and the other has the β coefficient equal to unity. In fact it may be verified immediately that

$$\begin{pmatrix}\beta+\gamma & \alpha \\ \epsilon & \delta\end{pmatrix} = \begin{pmatrix}\beta+0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1+\bar{\gamma} & \bar{\alpha} \\ \epsilon & \delta\end{pmatrix}$$

where $\bar{\gamma}(x\ \ y) = \gamma(x, y)/\beta(x)$, and $\bar{\alpha}(x) = \alpha(x)/\beta(x)$. If $\delta \neq 0$, the form of the transformation may be even further simplified, as follows:

$$(22)\qquad \begin{pmatrix}\beta+\gamma & \alpha \\ \epsilon & \delta\end{pmatrix} = \begin{pmatrix}\beta+0 & 0 \\ 0 & \delta\end{pmatrix}\begin{pmatrix}1+\bar{\gamma} & \bar{\alpha} \\ \bar{\epsilon} & 1\end{pmatrix}.$$

The transformation forming the first factor on the right side of (22), written in full in non-homogeneous coördinates, is

$$\phi'(x) = \frac{\beta(x)}{\delta}\phi(x),$$

that is, it consists in multiplication by an ordinary function. If we agree to call such a transformation a *functional multiplication*, we may state

THEOREM V: *A non-singular projective transformation of Type I, II, or III, is projectively equivalent to the product of a functional multiplication and a transformation of form*

$$\begin{pmatrix}1+\gamma & 0 \\ 0 & 1\end{pmatrix}, \qquad \begin{pmatrix}1+\gamma & \alpha \\ 0 & 1\end{pmatrix}, \qquad \textit{or} \qquad \begin{pmatrix}1+\gamma & 0 \\ \epsilon & 1\end{pmatrix},$$

respectively.

For δ cannot be zero when the transformation in question is reduced to its type form, since a non-singular transformation cannot be projectively equivalent to a singular transformation.

§ 9. FREDHOLM TRANSFORMATIONS.

According to Theorem V, every non-singular transformation of Type I is projectively equivalent to the product of a functional multiplication and a transformation of form

$$\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1 \end{pmatrix}.$$

This latter is, in non-homogeneous coördinates, the ordinary Fredholm transformation

$$\phi' = \phi + J\gamma\phi.$$

Conversely, if a projective transformation is projectively equivalent to the product of a functional multiplication and an ordinary Fredholm transformation, it must admit an invariant lineoid and an invariant point not on it. Hence

THEOREM VI: *A necessary and sufficient condition that a non-singular projective transformation be projectively equivalent to the product of a functional multiplication and an ordinary Fredholm transformation is that it admit an invariant lineoid and an invariant point which are not conjoint.*

COROLLARY: *Every non-singular symmetric projective transformation is projectively equivalent to an ordinary Fredholm transformation and multiplication by a constant.*

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